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Exact classical solutions of nonlinear sigma models on supermanifolds

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Abstract

We study two-dimensional nonlinear sigma models with target spaces being the complex super Grassmannian manifolds, that is, coset supermanifolds $G(m,p|n,q) \cong U(m|n)/[U(p|q) \otimes U(m-p|n-q)]$ for $0 \le p \le m$, $0 \le q \le n$ and $1 \le p+q$. The projective superspace $\mathbf{CP}^{m-1|n}$ is a special case of $p=1$, $q=0$. For the two-dimensional Euclidean base space, a wide class of exact classical solutions (or harmonic maps) are constructed explicitly and elementarily in terms of Gramm-Schmidt orthonormalisation procedure starting from holomorphic bosonic and fermionic supervector input functions. The construction is a generalisation of the non-super case published more than twenty years ago by one of the present authors.

Keywords: nonlinear sigma model on supermanifolds; Gramm-Schmidt orthonormalisation procedure; super Grassmannian manifold.

1 Introduction

A wide class of exact classical solutions are constructed in this paper, for Euclidean two-dimensional non-linear sigma models on complex super Grassmannian manifolds. The general motivation/background of the present work is the recent interest in 2-D non-linear sigma models on supergroups, in particular $PSU(1,1|2)$, $PSU(2,2|4)$ and more generally $PSL(n|n)$ [1] and some of their supercoset manifolds. They are related to superstrings propagating on certain AdS backgrounds. These models have also found applications in statistical mechanics, such as the integer quantum Hall effect and its recent generalisation, fermions with quenched disorder, percolation, polymers etc [2]. Like their non-super counterparts, these 2-D sigma models are classically integrable and enjoy an infinite number of local/non-local conservation laws [3]. In contrast to the non-super 2-D sigma models for which masses are dynamically generated by quantum effects, exact conformal invariance is preserved at the quantum level in some special supergroup sigma models [1]. This would mean, as in higher $\mathcal{N}$ supersymmetric gauge theories, that the quantum theory and classical theory are closely related and that quantum results could be inferred/derived from their relatively well-understood classical counterparts. In fact, various integrable structures and methods, for example, an infinite number of local/non-local conserved quantities, symmetry transformations, etc., have much simpler forms at the classical level than the quantum ones. A naive hope arises that exact and fairly general classical solutions, if available, could elucidate various aspects of the corresponding quantum field theory.

The present paper is a modest attempt into that general direction by providing a fairy wide class of exact solutions for non-linear sigma models on certain supercosets, namely the complex super Grassmannian. These are the supermanifold (not spacetime supersymmetric) versions of the complex projective space $\mathbf{CP}^{N-1}$ and complex Grassmannian sigma models [4]. Together with the sine-Gordon theory and $O(n)$ sigma models, they have been investigated quite extensively as a theoretical laboratory for four dimensional gauge theories for about a quarter of a century. For the complex Grassmannian $G(N,m) \cong U(N)/[U(m) \times U(N-m)]$ sigma models (including the $\mathbf{CP}^{N-1}$ [5] as a special case) on 2-D Euclidean space, quite general classical solutions (or harmonic maps in mathematics) were constructed by one of the present authors more than twenty years ago [6]. Starting from m holomorphic input vector functions, Gramm-Schmidt orthonormalisation procedure is applied to produce N unit column vectors of the corresponding $U(N)$ group. Certain subsets of these unit vectors constitute solutions of the complex Grassmannian sigma model. Although the structure of the super Grassmannian $G(m,p|n,q) \cong U(m|n)/[U(p|q) \otimes U(m-p|n-q)]$ is much more complicated than the non-super $G(N,m) \cong U(N)/[U(m) \times U(N-m)]$, the basic strategy of solution construction is about the same. One starts with p bosonic and q fermionic holomorphic input supervector functions and orthonormalise them in terms of Gramm-Schmidt procedure to produce $m+n$ basis vectors of the super unitary group $U(m|n)$. Again certain subsets of these unit supervectors constitute solutions of the complex super Grassmannian sigma model.

The paper is organised as follows. In section two, basic concepts and notation for supernumbers, supervectors, supervector spaces, supermatrices and super Grassmannians are introduced. The two-dimensional non-linear sigma models on the super Grassmannian manifolds are introduced in section three. The gauge invariant action, equations of motion in various equivalent forms are derived and symmetry properties are explored. Exact solutions are constructed in section four starting from p bosonic and q fermionic input supervectors. Gramm-Schmidt orthonormalisation procedure is applied to produce the unit supervectors of the super unitary group $U(m|n)$. The final section is for comments and a summary. The Appendix provides an elementary proof of one important formula (4.28) in section four.

2 Supervector space and super Grassmannian

Let us start with a few words on the Grassmann algebra and the supernumbers [7]. Here we use the standard notion of the Grassmann algebra Λ_N over $\mathbb{C}$, generated by ξ^a, $a = 1, \ldots$, N, which anticommute

$$\xi^a \xi^b = -\xi^b \xi^a, \quad (\xi^a)^2 = 0, \quad \forall a, b.$$

To be more precise, we use the inductive limit of $N \to \infty$, Λ_∞. The elements of Λ_∞ are called supernumbers. Every supernumber z has its body and soul

$$z = z_B + z_S,$$

where the body (z_B) is the ordinary complex number and the soul (z_S) vanishes when all the Grassmann generators are put to zero, $\xi^a \to 0$. A supernumber z has the inverse z^{-1} if and only if its body is non-vanishing $z_B \neq 0$.

Let us fix four non-negative integers m, n, p and q such that $2 \leq m+n$, $0 \leq p \leq m$, $0 \leq q \leq n$ and $1 \leq p+q$. Let $\mathbf{V}$ be a $\mathbb{Z}_2$-graded $(m+n)$-dimensional complex vector space (or supervector space of type (m,n) [7]) with the pure basis $\{e_i | i = 1, \ldots m+n\}$. Here m is the number of even basis vectors and n is the number of odd bases. It should be emphasised that the assignment of the $\mathbb{Z}_2$-grading to each index j is completely arbitrary. One can choose the indices of even grading $\mathbb{E}$ and odd grading $\mathbb{O}$ at will:

$$[e_j] = 0 \quad \text{for } j \in \mathbb{E}, \quad \#\mathbb{E} = m, \qquad [e_j] = 1 \quad \text{for } j \in \mathbb{O}, \quad \#\mathbb{O} = n. \tag{2.1}$$

Once the sets $\mathbb{E}$ and $\mathbb{O}$ are specified they must be kept fixed. One simple and often used choice is $\mathbb{E} = \{1, 2, \ldots, m\}$ and $\mathbb{O} = \{m+1, \ldots, m+n\}$. Then any supervector $\mathbf{a} \in \mathbf{V}$ is expanded in terms of the basis, $\mathbf{a} = \sum_i e_i\, a_i$, and it can be represented by an $(m+n)$-component column supervector,

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix}, \quad a_j \in \Lambda_\infty, \quad j = 1, \ldots, m+n. \tag{2.2}$$

A supervector with definite grading is called a pure vector. There are bosonic and fermionic pure vectors. A bosonic (resp. fermionic) pure vector $\mathbf{a} \in \mathbf{V}$ has gradings: $[a_i] = 0$ for $i \in \mathbb{E}$ and $[a_i] = 1$ for $i \in \mathbb{O}$ (resp. $[a_i] = 1$ for $i \in \mathbb{E}$ and $[a_i] = 0$ for $i \in \mathbb{O}$). In this paper we consider pure vectors only.

The hermitian conjugate of a supervector $\mathbf{a} \in \mathbf{V}$, denoted by $\mathbf{a}^\dagger$, is an $(m+n)$-component row supervector,

$$\mathbf{a}^\dagger = (a_1^*, \ldots, a_{m+n}^*).$$

The complex conjugation ($*$-operation) of supernumbers has the following properties [8],

$$(ab)^* = b^* a^*, \qquad (a^*)^* = a, \qquad [a^*] = [a].$$

The supervector space $\mathbf{V}$ can be endowed with an inner product: $\mathbf{V} \times \mathbf{V} \mapsto \Lambda_\infty$, denoted by

$$\langle \omega, \mathbf{v} \rangle \in \Lambda_\infty, \quad \omega, \mathbf{v} \in \mathbf{V}. \tag{2.3}$$

It is bilinear in following sense

$$\begin{aligned} &\langle \omega_1 a_1 + \omega_2 a_2, \mathbf{v}_1 b_1 + \mathbf{v}_2 b_2 \rangle \\ &\quad = a_1^* \langle \omega_1, \mathbf{v}_1 \rangle b_1 + a_1^* \langle \omega_1, \mathbf{v}_2 \rangle b_2 + a_2^* \langle \omega_2, \mathbf{v}_1 \rangle b_1 + a_2^* \langle \omega_2, \mathbf{v}_2 \rangle b_2 \end{aligned} \tag{2.4}$$

for $\omega_1, \omega_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$ and any supernumbers a_j and b_j, and enjoys,

$$\langle \mathbf{v}, \omega \rangle = \langle \omega, \mathbf{v} \rangle^*. \tag{2.5}$$

Let the pure basis $\{e_i | i = 1, \ldots, m+n\}$ be orthonormal. Then the inner product of two supervectors $\mathbf{a}_1$ and $\mathbf{a}_2$ can be expressed as

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \mathbf{a}_1^\dagger \mathbf{a}_2. \tag{2.6}$$

The norm of a supervector $\mathbf{a}$, denoted by $\|\mathbf{a}\|$, is defined by

$$\|\mathbf{a}\|^2 = \langle \mathbf{a}, \mathbf{a} \rangle = \mathbf{a}^\dagger \mathbf{a}, \tag{2.7}$$

which is an even supernumber. It is positive definite so long as the body $\mathbf{a}_B$ is non-vanishing. In this case we can define a normalised (unit) vector

$$\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|, \quad \mathbf{a}_B \neq \mathbf{0}. \tag{2.8}$$

It is easy to see that the body of a normalised vector is normalised, too:

$$\mathbf{u}_B = \mathbf{a}_B/\|\mathbf{a}_B\|, \quad \|\mathbf{u}_B\| = 1. \tag{2.9}$$

Let us introduce a new basis by a linear combination of the old one:

$$e'_j = \sum_l e_l \, U_{l\,j}, \quad j = 1, \ldots, m+n. \tag{2.10}$$

By requiring that the new basis is again orthonormal, we obtain the condition

$$\sum_l U^*_{l\,j} \, U_{l\,k} = \delta_{j\,k}, \quad \text{or} \quad U^\dagger U = 1_{m+n}, \tag{2.11}$$

in which 1_{m+n} is the $(m+n) \times (m+n)$ identity matrix. Namely U is a unitary supermatrix. Since the new basis vectors are also required to be pure with the same grading as the old one, there are m even and n odd basis vectors. The grading of $U_{l\,j}$ is constrained as

$$[U_{l\,j}] = [l] + [j], \quad \text{mod } 2. \tag{2.12}$$

That is each column supervector of U

$$U = (\mathbf{u}_1, \ldots, \mathbf{u}_{m+n}), \qquad U \in U(m|n), \tag{2.13}$$

is either bosonic or fermionic. In other words, there are m bosonic and n fermionic column supervectors satisfying the orthonormality condition

$$\mathbf{u}_j^\dagger \mathbf{u}_k = \delta_{j\,k}, \quad j,k = 1,\ldots,m+n, \tag{2.14}$$

as rephrased from (2.11).

Here are some facts and notation for $(m+n)\times(m+n)$ supermatrices used in this paper. Among the $m+n$ indices of a supermatrix M, m are even $[i]=0$ and n are odd $[i]=1$ and the $\mathbb{Z}_2$-grading of the indices are fixed once and for all as in (2.1). A supermatrix M is called even, $[M]=0$, if $[M_{(e)\,(e)}]=[M_{(o)\,(o)}]=0$ and $[M_{(e)\,(o)}]=[M_{(o)\,(e)}]=1$; whereas, a supermatrix M is called odd, $[M]=1$, if $[M_{(e)\,(e)}]=[M_{(o)\,(o)}]=1$ and $[M_{(e)\,(o)}]=[M_{(o)\,(e)}]=0$. Here (e) stands for the indices from $\mathbb{E}$ and (o) from $\mathbb{O}$. These are called supermatrices of definite grading and we will consider such supermatrices only in this paper. The supertrace of a supermatrix M is defined by [8]

$$\mathrm{str}\,(M) = \sum_{j=1}^{m+n} (-1)^{([j]+[M])[j]}\, M_{j\,j}. \tag{2.15}$$

The hermitian conjugate of M, denoted by $M^\dagger$, is an $(m+n)\times(m+n)$ supermatrix with the entries

$$\left(M^\dagger\right)_{ij} = M^*_{ji}, \qquad i,j = 1,\ldots,m+n.$$

The supertrace defined by (2.15) enjoys the following properties,

$$\mathrm{str}\left(M^\dagger\right) = \mathrm{str}\,(M)^*, \tag{2.16}$$

$$\mathrm{str}\,(M_1 M_2) = (-1)^{[M_1][M_2]}\,\mathrm{str}\,(M_2 M_1)\,. \tag{2.17}$$

The ordinary complex Grassmannian manifold $G(N,p)$ is a collection of p-dimensional sub-vector spaces within a complex N-dimensional vector space $\mathbb{C}^N$. A point in $G(N,p)$ is specified by a choice of p-orthonormal basis vectors $\{e''_j\}$, $j=1,\ldots,p$ which is a subset of an orthonormal basis $\{e'_j\}$, $j=1,\ldots,N$ obtained by an arbitrary unitary transformation $(U(N))$ from a fixed orthonormal basis $\{e_j\}$, $j=1,\ldots,N$. Any unitary transformations among the chosen vectors $\{e''_j\}$, $j=1,\ldots,p$, $(U(p))$ and among the not-chosen vectors $\{e''_j\}$, $j=p+1,\ldots,N$, $(U(N-p))$ are immaterial. Thus we have

$$G(N,p) = \frac{U(N)}{U(p)\times U(N-p)}. \tag{2.18}$$

In this paper we discuss the complex Grassmannian supermanifold $G(m,p|n,q)$, which consists of a collection of sub-supervector spaces of (p,q) type within a complex supervector space $\mathbf{V}$ of (m,n) type. A point in $G(m,p|n,q)$ is specified by a choice of $(p+q)$-orthonormal basis vectors $\{e''_j\}$, $j=1,\ldots,p+q$ among which p are even and q are odd. It is a subset of an orthonormal basis $\{e'_j\}$, $j=1,\ldots,m+n$ obtained by an arbitrary super unitary transformation $(U(m|n))$ from a fixed orthonormal basis $\{e_j\}$, $j=1,\ldots,m+n$. Any super unitary transformations among the chosen vectors $\{e''_j\}$, $j=1,\ldots,p+q$, $(U(p|q))$ and among the not-chosen vectors $\{e''_j\}$, $j=p+q+1,\ldots,m+n$, $(U(m-p|n-q))$ are immaterial. Thus we have

$$G(m,p|n,q)=\frac{U(m|n)}{U(p|q)\times U(m-p|n-q)}. \tag{2.19}$$

It is a Riemannian symmetric superspace [9], as the ordinary $G(N,p)$ is a Riemannian symmetric space. It should be stressed that the $\mathbb{Z}_2$-grading of the new chosen basis $\{e''_j\}$, $j=1,\ldots,p+q$, is completely independent of the original basis $\{e_j\}$, $j=1,\ldots,m+n$ (2.1), since it refers to the $\mathbb{Z}_2$-grading of the new sub-supervector space of (p,q) type. A different choice of $\{e''_j\}$, $j=1,\ldots,p+q$ with a different $\mathbb{Z}_2$-grading corresponds to a different sub-supervector space.

3 Nonlinear sigma models on Supermanifolds

We shall study the two-dimensional nonlinear sigma model with the target space being this particular Riemannian symmetric superspace $G(m,p|n,q)$. The base space is the two-dimensional Euclidean space. The resulting sigma models are the supermanifold version of the ordinary $G(N,p)$ models considered in [4, 6].

Let $x=(x_1,x_2)\in\mathbb{R}^2$ be the coordinates of the two-dimensional Euclidean space and $g=g(x)$ be a field which takes value in $U(m|n)$. We decompose it into two parts

$$g=(X,Y), \tag{3.1}$$

with

$$X=(\mathbf{z}_1,\ldots,\mathbf{z}_{p+q}),\qquad Y=(\mathbf{z}_{p+q+1},\ldots,\mathbf{z}_{m+n}). \tag{3.2}$$

Here $\mathbf{z}_i$ is an $(m+n)$-component column supervector, either bosonic or fermionic. There are p bosonic and q fermionic column supervectors in X and $m-p$ bosonic and $n-q$ fermionic column supervectors in Y. As explained above, the $G(m,p|n,q)$ sigma model is described by the dynamical variable $X = X(x)$, satisfying the constraint

$$X^\dagger X = I_{p+q}, \tag{3.3}$$

originating from the unitarity (2.11) of g. The Grassmannian structure of $G(m,p|n,q)$ is incorporated through the covariant derivative for X,

$$D_\mu X = \partial_\mu X - X A_\mu, \qquad \mu = 1,2, \tag{3.4}$$

where the gauge potential A_μ is given by

$$A_\mu = X^\dagger \partial_\mu X, \qquad \mu = 1,2. \tag{3.5}$$

The constraints (3.3) imply that the gauge potential satisfies

$$(A_\mu)^\dagger = -A_\mu, \qquad \mu = 1,2. \tag{3.6}$$

The action of the $G(m,p|n,q)$ nonlinear sigma model in two-dimensional Euclidean space is given by

$$S = \int d^2x \,\mathrm{str}\left((D_\mu X)^\dagger \; (D_\mu X) \right), \tag{3.7}$$

where as usual the repeated indices mean the summation. This action has the $U(p|q)$ local gauge symmetry,

$$X(x) \longrightarrow X'(x) = X(x)\,h(x), \quad h(x) \in U(p|q), \tag{3.8}$$

as well as the global $U(m|n)$ symmetry

$$X(x) \longrightarrow X'(x) = g_0\,X(x), \quad g_0 \in U(m|n), \quad \partial_\mu g_0 = 0. \tag{3.9}$$

This is because the matrices X, $h(x)$ and g_0 are always even supermatrices, i.e., $[X] = [h(x)] = [g_0] = 0$, and the supertrace formula (2.17) applies without the extra sign. The classical equation of motion of the model is

$$D_\mu D_\mu X + X \; (D_\mu X)^\dagger \, D_\mu X = 0. \tag{3.10}$$

The model can also be defined in a gauge invariant way if we introduce a projection supermatrix P

$$P \equiv X\,X^\dagger = \sum_{j=1}^{p+q} \mathbf{z}_j \mathbf{z}_j^\dagger, \tag{3.11}$$

which is obviously gauge invariant under (3.8) and has rank of $p+q$ (p bosonic eigenvectors and q fermionic eigenvectors), and enjoys the properties,

$$P^\dagger = P = P^2. \tag{3.12}$$

The action (3.7) can be re-expressed in terms of the projection supermatrix

$$S = \frac{1}{2}\int d^2x\, \mathrm{str}\left(\partial_\mu P\, \partial_\mu P\right), \tag{3.13}$$

and the corresponding equation of motion becomes

$$[\partial_\mu\partial_\mu P,\, P] = 0. \tag{3.14}$$

The remaining part Y (3.2) of the unitary supermatrix g defines another projection supermatrix $\bar{P}$:

$$\bar{P} \equiv Y\, Y^\dagger = \sum_{j=p+q+1}^{m+n} \mathbf{z}_j \mathbf{z}_j^\dagger = 1_{m+n} - P, \tag{3.15}$$

which is orthogonal to P. It has rank $m+n-(p+q)$ (i.e., $m-p$ bosonic eigenvectors and $n-q$ fermionic eigenvectors). This projection supermatrix $\bar{P}$ satisfies the same equation of motion (3.14) as that of P, reflecting the obvious symmetry $\{p,q\} \leftrightarrow \{m-p, n-q\}$ of the super Grassmannian $G(m,p|n,q) \cong U(m|n)/[U(p|q)\otimes U(m-p|n-q)]$.

For the simplest case of $p=1$ and $q=0$, the super Grassmannian manifold $G(m,p|n,q)$ reduces to the projective superspace $\mathbf{CP}^{m-1|n}$ and the corresponding sigma model was investigated in [2].

4 Exact solutions

In this section, we shall construct a series of solutions of the super Grassmannian sigma model given by the action (3.7). These solutions are expressed in terms of a set of holomorphic bosonic and fermionic supervector input functions.

Let us introduce the complex coordinates of the two-dimensional Euclidean space

$$x_+ = x_1 + \mathrm{i}x_2, \quad x_- = x_1 - \mathrm{i}x_2. \tag{4.1}$$

The equation of motion (3.10) for X is rewritten as

$$D_+ D_- X + X\,(D_- X)^\dagger\,(D_- X) = 0, \quad \partial_\pm = \frac{\partial}{\partial x_\pm}, \tag{4.2}$$

or equivalently,

$$D_-D_+X + X\,(D_+X)^\dagger\,(D_+X) = 0. \tag{4.3}$$

Likewise, one may rewrite the gauge invariant equation (3.14) as

$$[\partial_+\partial_-P,\; P] = 0. \tag{4.4}$$

To construct generic solutions to the equation of motion, let us consider as input $(p+q)$ linearly independent *holomorphic pure* supervectors of (m,n) type, among which p are *bosonic* and q are *fermionic*, but the order is completely arbitrary, as mentioned above. Let us denote them

$$\mathbf{f}_1,\,\mathbf{f}_2,\ldots,\mathbf{f}_{p+q},\qquad \partial_-\mathbf{f}_j = 0,\quad j = 1,\ldots,p+q. \tag{4.5}$$

A different ordering of these $(p+q)$ input supervectors will give rise to different solutions. From these supervectors $\{\mathbf{f}_j\}$, $j = 1,\ldots,p+q$ we construct $m+n$ holomorphic pure supervectors of (m,n) type by successive differentiation with respect to x_+:

$$\begin{aligned}
\mathbf{f}_{p+q+1} &= \partial_+\mathbf{f}_1,\ \mathbf{f}_{p+q+2} = \partial_+\mathbf{f}_2,\ \ldots,\ \mathbf{f}_{2p+2q} = \partial_+\mathbf{f}_{p+q},\\
\mathbf{f}_{2p+2q+1} &= \partial_+^2\mathbf{f}_1,\ \mathbf{f}_{2p+2q+2} = \partial_+^2\mathbf{f}_2,\ldots,\mathbf{f}_{3p+3q} = \partial_+^2\mathbf{f}_{p+q},\\
\mathbf{f}_{3p+3q+1} &= \partial_+^3\mathbf{f}_1,\ \mathbf{f}_{3p+3q+2} = \partial_+^3\mathbf{f}_2,\ldots,\mathbf{f}_{4p+4q} = \partial_+^3\mathbf{f}_{p+q},\\
&\ldots,\quad \ldots,\quad \mathbf{f}_{m+n}
\end{aligned} \tag{4.6}$$

Supposing that the resulting supervectors $\mathbf{f}_1,\ldots,\mathbf{f}_{m+n}$ are linearly independent and their body vectors $\mathbf{f}_{1\,B},\ldots,\ \mathbf{f}_{m+n\,B}$ are also linearly independent, we apply the Gramm-Schmidt procedure to obtain an orthonormal basis of the superspace $\mathbf{V}$:

$$\mathbf{e}_1,\ldots,\mathbf{e}_{m+n},\quad \mathbf{e}_j^\dagger\mathbf{e}_k = \delta_{j\,k}. \tag{4.7}$$

To be more specific, the Gramm-Schmidt procedure goes as follows

$$\begin{array}{ll}
\mathbf{e}_1 = \frac{\mathbf{g}_1}{||\mathbf{g}_1||}, & \mathbf{g}_1 = \mathbf{f}_1,\\
\mathbf{e}_2 = \frac{\mathbf{g}_2}{||\mathbf{g}_2||}, & \mathbf{g}_2 = \mathbf{f}_2 - \mathbf{e}_1\,(\mathbf{e}_1^\dagger\,\mathbf{f}_2),\\
\quad\vdots & \qquad\vdots\\
\mathbf{e}_j = \frac{\mathbf{g}_j}{||\mathbf{g}_j||}, & \mathbf{g}_j = \mathbf{f}_j - \sum_{k=1}^{j-1}\mathbf{e}_k\,(\mathbf{e}_k^\dagger\,\mathbf{f}_j),\\
\quad\vdots & \qquad\vdots
\end{array} \tag{4.8}$$

It is easy to check that the resulting basis vectors $\mathbf{e}_i$ are pure vectors and the $\mathbb{Z}_2$-grading has a period $p+q$ because of the preparation of the supervectors $\{\mathbf{f}_j\}$ (4.6):

$$[\mathbf{e}_{p+q+j}] = [\mathbf{e}_j],\quad j = 1,\ldots,m+n-(p+q). \tag{4.9}$$

Here is one important remark concerning the Gramm-Schmidt orthonormalisation of the above supervectors (4.6). For the generic case of the chosen integers m, n, p and q, the above Gramm-Schmidt procedure does not come to the end $\mathbf{e}_{m+n}$ but it stops at $\mathbf{e}_N$, for certain $N \leq m+n$. There are m even and n odd basis vectors in $\mathbf{V}$. However, the orthonormalisation proceeds by the unit of p even and q odd supervectors and the choice of p and q is independent of m and n except for the obvious constraints $0 \leq p \leq m$, $0 \leq q \leq n$ and $1 \leq p+q$. Therefore it can generically happen that either the entire m even basis or the entire n odd basis is already made before all the vectors in (4.5), (4.6) can be orthonormalised by means of (4.8) to end with $\mathbf{e}_{m+n}$. Let $\mathbf{f}_{N+1}$ be the $(m+1)$-th even supervector or the $(n+1)$-th odd supervector in (4.6) to be orthonormalised. Then its projection

$$\mathbf{g}_{N+1} = \left(1_{m+n} - \sum_{j=1}^{N} \mathbf{e}_j \mathbf{e}_j^\dagger\right) \mathbf{f}_{N+1} \tag{4.10}$$

cannot have a non-vanishing body and therefore cannot be normalised. If it has, $\mathbf{g}_{N+1}$ can be normalised to obtain the $(m+1)$-th even base or $(n+1)$-th odd base, which cannot happen in a supervector space $\mathbf{V}$ of (m,n) type. To summarise, the set of orthonormal supervectors obtained by the above procedure (4.8) will be

$$\{\mathbf{e}_j \mid \ j = 1, 2, \ldots, \min(N, m+n)\}. \tag{4.11}$$

Hereafter we use the notation N as meaning $\min(N, m+n)$ for simplicity.

At first sight this might seem rather strange. But at closer inspection, it turns out to be rather natural. For instance, consider the extreme case of $q = 0$. In this case we start with only the bosonic input supervectors. The orthonormalisation produces the bosonic base vectors only, and it stops at $\mathbf{e}_m$. This phenomenon of intermediate stopping of the orthonormalisation procedure does not happen in the non-super Grassmannian sigma models [6].

By picking up $p+q$ *consecutive orthonormal supervectors*, we define the following $(m+n) \times (p+q)$ matrices:

$$\begin{aligned} X_{(1)} &= (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{p+q}), \\ X_{(2)} &= (\mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_{p+q+1}), \\ \vdots \qquad & \qquad \vdots \\ X_{(N-p-q+1)} &= (\mathbf{e}_{N-p-q+1}, \ldots, \mathbf{e}_N). \end{aligned} \tag{4.12}$$

All satisfy the constraint (3.3) and they contain p even and q odd basis supervectors. But the order of the even and the odd basis is not the same. Then $X_{(j)}$, $j = 1, \ldots, N-p-q+1$

and their gauge transformed form $X'_{(j)} = X_{(j)}h_{(j)}$ *satisfy the equation of motion* (4.2). The proof is quite elementary.

Let us start the proof with the analytic property of the orthonormal supervectors. Because of the orthonormalisation procedure, we know that each basis vector $\mathbf{e}_i$ can be expressed in terms of $\{\mathbf{f}_j\}$, $j = 1,\ldots, i$:

$$\mathbf{e}_i = \sum_{j=1}^{i} \mathbf{f}_j \, a_j, \qquad i = 1, \ldots, N, \tag{4.13}$$

where the coefficients $\{a_j\}$ are supernumbers. The above expression implies that the expansion of each vector $\mathbf{f}_i$ in terms of the basis $\{\mathbf{e}_i\}$ has the same triangular form

$$\mathbf{f}_i = \sum_{j=1}^{i} \mathbf{e}_j \, b_j, \qquad i = 1, \ldots, N, \tag{4.14}$$

with some supernumbers $\{b_j\}$. By differentiating (4.13) with respect to x_-, we have $\partial_- \mathbf{e}_i = \sum_{j=1}^{i} \mathbf{f}_j \, \partial_- a_j$ since $\{\mathbf{f}_j\}$ are holomorphic supervectors. Using the expansion (4.14), we find that

$$\partial_- \mathbf{e}_i = \sum_{j=1}^{i} \mathbf{e}_j \, (\mathbf{e}_j^\dagger \, \partial_- \mathbf{e}_i), \quad i = 1, \ldots, N. \tag{4.15}$$

Moreover, by the ways of constructing the vectors $\{\mathbf{f}_1, \ldots, \mathbf{f}_{m+n}\}$ (4.6) we have the following useful expansions

$$\partial_+ \mathbf{e}_i = \sum_{j=1}^{i+p+q} \mathbf{e}_j \, (\mathbf{e}_j^\dagger \partial_+ \mathbf{e}_i), \qquad 1 \le i \le N - (p+q). \tag{4.16}$$

The above two relations (4.15)-(4.16) will play an essential role in the proof for the solutions (4.12).

We will show that

$$X_{(j)} = (\mathbf{e}_j, \mathbf{e}_{j+1}, \ldots, \mathbf{e}_{j+p+q-1}), \quad j = 1, 2, \ldots, N - (p+q) + 1, \tag{4.17}$$

solves the equation of motion (4.2), or the corresponding projection supermatrix

$$P_{(j)} = X_{(j)} \, X_{(j)}^\dagger = \sum_{k=j}^{j+p+q-1} \mathbf{e}_k \mathbf{e}_k^\dagger, \quad P_{(j)}^\dagger = P_{(j)} = P_{(j)}^2, \tag{4.18}$$

satisfies (4.4):

$$[\partial_+ \partial_- P_{(j)}, P_{(j)}] = 0. \tag{4.19}$$

Although proper care is needed for the grading problem, most formulas formally look essentially the same as those in the non-super case [6]. We will proceed in a similar way as those in the non-super case. Following [6], we introduce an auxiliary supermatrix variable $Q_{(j)}$ by

$$Q_{(j)} = \sum_{k=1}^{j-1} \mathbf{e}_k \mathbf{e}_k^\dagger, \qquad j = 1, 2, \ldots, N - (p+q) + 1, \tag{4.20}$$

which is a projection supermatrix, too:

$$Q_{(j)}^\dagger = Q_{(j)} = Q_{(j)}^2. \tag{4.21}$$

It is of rank $j-1$ and is orthogonal to $P_{(j)}$

$$P_{(j)} Q_{(j)} = Q_{(j)} P_{(j)} = 0. \tag{4.22}$$

Hereafter the suffix (j) of $P_{(j)}$ and $Q_{(j)}$ is fixed and will not be written explicitly to make the notation simple. From the x_- derivative relation (4.15) we obtain

$$(\partial_- Q) Q = 0, \tag{4.23}$$

and

$$\partial_-(P+Q)(P+Q) = 0. \tag{4.24}$$

Taking the x_- derivative of the orthogonality relation (4.22), we obtain

$$(\partial_- P) Q + P(\partial_- Q) = 0. \tag{4.25}$$

Another simple consequence of the x_- derivative relation (4.15) is

$$P(\partial_- Q) = 0. \tag{4.26}$$

By combining (4.23)–(4.26), we obtain a simple relationship

$$(\partial_- P) P + (\partial_- Q) P = 0. \tag{4.27}$$

Next we consider the consequences of the x_+ derivative relation (4.16). The only essential formula is

$$P(\partial_+ Q) = \partial_+ Q. \tag{4.28}$$

This reflects the facts that the x_+ differentiation sends $\mathbf{f}_k$ to $\mathbf{f}_{k+p+q}$ (4.6) and that the projection supermatrix P consists of just the $p+q$ orthonormal supervectors after Q. We will give a simple proof in the Appendix. The hermitian conjugation of the above formula reads

$$(\partial_- Q)P = \partial_- Q. \tag{4.29}$$

Equation (4.27) and (4.29) combine to give

$$(\partial_- P)P + \partial_- Q = 0 \tag{4.30}$$

together with its hermitian conjugation

$$P(\partial_+ P) + \partial_+ Q = 0. \tag{4.31}$$

By subtracting the x_- derivative of (4.31) from the x_+ derivative of (4.30), we obtain the desired formula

$$[\partial_+ \partial_- P, P] = 0, \tag{4.32}$$

which completes the proof.

5 Comments and Summary

Some comments and remarks are in order. The first is about *instanton* and *anti-instanton solutions*. As is obvious from (4.2) and (4.3), if X satisfies

$$D_- X = 0, \qquad \text{or} \qquad D_+ X = 0 \tag{5.1}$$

then the full equation of motion is trivially satisfied. These are simply the covariant version of the equations characterising the holomorphic and anti-holomorphic functions

$$\partial_- f = 0, \qquad \text{or} \qquad \partial_+ f = 0.$$

In analogy with the solutions of the (anti-)self-dual equations for the gauge field strength, which automatically satisfy the full second order equations, these solutions are called *instanton* and *anti-instanton solutions*, respectively. In terms of the projector supermatrix P, the first order equations (5.1) are written as

$$(\partial_- P)P = 0, \qquad \text{or} \qquad P(\partial_- P) = 0. \tag{5.2}$$

By differentiating the first equation with respect to x_+, we obtain

$$(\partial_+ \partial_- P)P + (\partial_- P)(\partial_+ P) = 0, \tag{5.3}$$

and its hermitian conjugation

$$P(\partial_+\partial_-P) + (\partial_-P)(\partial_+P) = 0. \tag{5.4}$$

By subtracting these two equations, we obtain $[\partial_+\partial_-P, P] = 0$. Thus the full equation of motion follows if either of (5.2) is satisfied. Among our explicit solutions, the first one, $X_{(1)}$, which is obtained from the input supervectors only without any differentiation, is the instanton solution. For $j = 1$, $Q_{(j)} = 0$ and (4.27), $(\partial_-P)P + (\partial_-Q)P = 0$, simply means $(\partial_-P_{(1)})P_{(1)} = 0$. As is easily expected the anti-instanton is the last one, $X_{(m+n-p-q+1)}$. For $j = m + n - p - q + 1$, one has $Q_{(j)} + P_{(j)} = 1_{m+n}$. Thus (4.26), $P(\partial_-Q) = 0$ means $P_{(j)}(\partial_-P_{(j)}) = 0$. As remarked earlier, our orthonormalisation procedure might not come to the end and the anti-instanton might not be included in our set of solutions. It may seem that our solution generation method discriminates the anti-instantons over instantons but the situation could be reversed if we decide to use the anti-holomorphic supervectors as input.

A few words on other types of solutions. The solutions explored in section 4 are called *generic*, since they depend on the maximal number of input data for $G(m, p|n, q)$, p bosonic and q fermionic holomorphic supervectors. A closer look at the proof might reveal that the same construction method with less input holomorphic bosonic (fermionic) supervectors also produces solutions of the super Grassmannian model $G(m, p|n, q)$. They are called degenerate solutions after the non-super case [6]. Furthermore, one may also construct some reducible solutions in the sense of [6] from these resulting degenerate ones as those in the non-super case. The completeness of the solutions thus obtained is beyond the scope of the present paper.

As is well known, any solution of the super Grassmannian $G(m, p|n, q) \cong U(m|n)/[U(p|q) \otimes U(m - p|n - q)]$ sigma model for various p and q provides a very special class of solutions of the supergroup (or the principal chiral) $U(m|n)$ sigma model. Take

$$g = 1_{m+n} - 2P_{(j)}, \qquad j = 1, \ldots, N - (p + q) + 1,$$

for any j and for arbitrary p and q. It is a special element of $U(m|n)$ satisfying the condition $g^2 = 1_{m+n}$ and the equation of motion:

$$\partial_\mu(g^{-1}\partial_\mu g) = -2\left[\partial_\mu\partial_\mu P_{(j)}, P_{(j)}\right] = 0, \tag{5.5}$$

thanks to the projection properties of $P_{(j)}$ (4.18).

Here is a summary. We have formulated non-linear sigma models on certain supermanifolds, the complex super Grassmannian $G(m,p|n,q)$ including the super projective spaces $\mathbf{CP}^{m-1|n}$. The base space is the two-dimensional Euclidean space. These are massless scalar field theories with non-linear geometrical constraints due to the supermanifolds. A wide class of classical exact solutions, or *harmonic maps* are constructed explicitly and elementarily in terms of the Gramm-Schmidt orthonormalisation procedure starting from the input holomorphic supervectors of type (m,n), among them p bosonic and q fermionic.

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Appendix: Proof of (4.28)

Here we provide a straightforward proof of (4.28), which is essential for the construction of solutions. Using the relation (4.15)-(4.16), we evaluate $\partial_+ Q$:

$$
\begin{aligned}
\partial_+ Q \quad &= \quad \sum_{l=1}^{j-1}\Big((\partial_+\mathbf{e}_l)\,\mathbf{e}_l^\dagger + \mathbf{e}_l\,\partial_+\mathbf{e}_l^\dagger\Big) = \sum_{l=1}^{j-1}\Big((\partial_+\mathbf{e}_l)\,\mathbf{e}_l^\dagger + \mathbf{e}_l\,(\partial_-\mathbf{e}_l)^\dagger\Big)\\
&\overset{(4.15)}{=} \quad \sum_{l=1}^{j-1}\left[(\partial_+\mathbf{e}_l)\,\mathbf{e}_l^\dagger + \mathbf{e}_l\sum_{k=1}^{l}\Big(\mathbf{e}_k\,\mathbf{e}_k^\dagger\partial_-\mathbf{e}_l\Big)^\dagger\right]\\
&\overset{(4.16)}{=} \quad \sum_{l=1}^{j-1}\left[\sum_{k=1}^{l+p+q}\mathbf{e}_k\left(\mathbf{e}_k^\dagger\partial_+\mathbf{e}_l\right)\mathbf{e}_l^\dagger + \mathbf{e}_l\sum_{k=1}^{l}\left(\partial_+\mathbf{e}_l^\dagger\,\mathbf{e}_k\right)\mathbf{e}_k^\dagger\right]\\
&= \quad \sum_{l=1}^{j-1}\sum_{k=1}^{l+p+q}\mathbf{e}_k\left(\mathbf{e}_k^\dagger\partial_+\mathbf{e}_l\right)\mathbf{e}_l^\dagger - \sum_{k=1}^{j-1}\sum_{l=1}^{k}\mathbf{e}_k\left(\mathbf{e}_k^\dagger\,\partial_+\mathbf{e}_l\right)\mathbf{e}_l^\dagger. \qquad \text{(A.1)}
\end{aligned}
$$

Among the first summation terms, we decompose

$$
\sum_{k=1}^{l+p+q}\mathbf{e}_k = \sum_{k=1}^{j-1}\mathbf{e}_k + \sum_{k=j}^{l+p+q}\mathbf{e}_k.
$$

The second sum is annihilated if multiplied by $(1-P)$ from the left and we obtain

$$\begin{aligned}(1-P)\partial_+ Q &= (1-P)\left[\sum_{l=1}^{j-1}\sum_{k=1}^{j-1} - \sum_{k=1}^{j-1}\sum_{l=1}^{k}\right] \mathbf{e}_k \left(\mathbf{e}_k^\dagger \partial_+ \mathbf{e}_l\right) \mathbf{e}_l^\dagger \\ &= (1-P)\sum_{k=1}^{j-1}\sum_{l=k+1}^{j-1} \mathbf{e}_k \left(\mathbf{e}_k^\dagger \partial_+ \mathbf{e}_l\right) \mathbf{e}_l^\dagger \\ &= -(1-P)\sum_{k=1}^{j-1}\sum_{l=k+1}^{j-1} \mathbf{e}_k \left((\partial_- \mathbf{e}_k)^\dagger \mathbf{e}_l\right) \mathbf{e}_l^\dagger = 0. \end{aligned} \tag{A.2}$$

The last equality is due to (4.15). Thus we obtain $P(\partial_+ Q) = \partial_+ Q$ and (4.28) is proved.

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